Inflight Observability of Gyro Reference Frame Misalignment

Whittak H. Huang*
Martin Marietta Corporation, Denver, Colorado

The significance of this paper is the in-depth examination of the effect of the gyro misalignment on the attitude knowledge error in a strapdown environment. A rigorous formulation of the attitude error in terms of a direction cosine matrix is presented and the equation of motion is derived as a matrix function of the attitude angular rate vector. Two theorems are stated: The first shows that the attitude error, induced by the initial attitude knowledge error and the gyro alignment error, is a function of the attitude only and, thus, is independent of the attitude trajectory. It is also shown that the two errors, the initial attitude error and the misalignment, which behave similarly, are definitely different. The second theorem shows that with an infinitesimal field-of-view line-of-sight (LOS) sensor, the initial attitude knowledge error and the gyro misalignment with respect to the LOS sensor is not separately observable. Other interesting consequences of these theorems are pointed out as well.

Nomenclature

a_1, a_1', etc.	= unit vectors
$Ce(t)$	= attitude knowledge error DCM
$Cg(t)$	= G-frame attitude DCM at time t
$Cgk(t)$	= gyro-propagated attitude knowledge
Cm	= constant misalignment DCM from S frame to G frame
$Cs(t)$	= S-frame attitude DCM at time t
$Cx(t)$	= intermediate error DCM
DCM	= direct cosine matrix
$e(t)$	= small attitude error expressed in rotational vector
e_x, e_y, e_z	= components of $e(t)$ in S frame
G frame	= gyro reference frame
I	= identity matrix (3×3)
k_1, k_1', etc.	= components of m projected onto a_1, a_2, a_1', a_2', a_3
LOS	= line of sight
m	= small misalignment error expressed in rotational vector
m_x, m_y, m_z	= components of m in S frame
s, s'	= star unit vectors in S frame
S frame	= LOS sensor reference frame
t, t_1, t_2	= times
T_1, T_2	= dimension reduction transformation (3×2)
$\tilde{T}_1$, $\tilde{T}_2$	= transpose of T_1, T_2, respectively
v	= LOS boresight unit vector
Wg	= inertial angular rate vector in G frame
Ws	= inertial angular rate vector in S frame

I. Introduction

ONE of the elements vital to the application of the strapdown gyroscope in the attitude determination design is the update measurement. The update measurement relates the gyro-propagated attitude to the main instrument that the system is designed for. In many cases, especially for celestial instrument, it serves to reset the gyro-propagated attitude and to prevent it from eventually diverging away from the true attitude. Generally, this divergence comes about because of inherent mechanical and electrical inaccuracies or instabilities in the hardware gyroscope, such as torquer scale-factor error, torquer/mounting misalignment, g-sensitive effects, and bias drift. Understanding the effects of the gyro hardware error on the propagated attitude is vital in minimizing the divergence, as one may be able to design calibration schemes that would precisely determine these hardware parameter errors.

This paper analyzes the effect of the gyro misalignment on the attitude knowledge error in a strapdown environment. It embarks on a rigorous thesis to show certain important properties of the attitude error due mainly to the initial attitude knowledge error and gyro reference frame misalignment with respect to an absolute reference frame in the system.

Section II defines the subject attitude knowledge and derives the equation of motion, considering only the initial attitude knowledge error and the misalignment. These two error sources, although similar in nature, are definitely different as shown. Section III presents the closed-form solution to the equation of motion and proves an important theorem concerning the error induced by the initial attitude knowledge error and misalignment between attitude updates. Section IV looks into the observability of misalignment in an ideal update environment where three axes of attitude knowledge error can be measured simultaneously. Section V discusses the misalignment observability in an infinitesimal field of view LOS sensor update design. Section VI concludes with the significance of this thesis.

II. Attitude Knowledge Error Formulation

A variety of attitude representations are available. For the convenience of mathematical proofs, direction cosine matrices are used in this paper.[1]

Let the coordinate misalignment DCM be denoted as Cm, such that

$$Cg = Cm * Cs \tag{1}$$

where Cg and Cs are the attitude DCM of the gyro reference frame (G) and the LOS sensor (S) reference frame as shown in Fig. 1.

Received June 23, 1986; presented as Paper 86-2151 at the AIAA Guidance, Navigation and Control Conference, Williamsburg, VA, Aug. 18-20, 1986; revision received June 15, 1987.

*Senior Group Engineer, Denver Aerospace Division, Guidance and Control Section, Magellan AACS Analysis Work Unit Manager. Member AIAA.

The knowledge of Cg, denoted as Cgk, may be in error from the true attitude Cs because of the inaccuracies of the gyro hardware parameters such as scale factors, misalignment, the bias drift, or the initial attitude knowledge. The subject of this paper deals with the effect of the misalignment Cm. Let the attitude knowledge error be the DCM $Ce(t)$ at time t, then

$$Cgk(t) = Ce(t)^* Cs(t) \tag{2}$$

Since the attitude knowledge $Cgk(t)$ is propagated using the equation of motion[1]

$$\frac{\mathrm{d}}{\mathrm{d}}\frac{Cgk(t)}{t} = \{Wg\}^* Cgk(t) \tag{3}$$

and the true attitude of the S frame satisfies the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}}\frac{Cs(t)}{t} = \{Ws\}^* Cs(t) \tag{4}$$

where the curry bracket $\{\}$ defines an skew symmetric 3×3 matrix using the three components of the vector inside the bracket, i.e.,

$$\{Ws\} = \begin{bmatrix} 0 & Wsz & -Wsy \\ -Wsz & 0 & -Wsx \\ Wsy & -Wsx & 0 \end{bmatrix} \tag{5}$$

The vectors Wg and Ws are the rotational rate vectors of the body measured in the G frame (i.e., by the gyro) and in the S frame, respectively, and are related to by the equation

$$Wg = Cm\,Ws \tag{6}$$

The time derivative of Eq. (2) yields

$$\frac{\mathrm{d}}{\mathrm{d}}\frac{Cgk(t)}{t} = \frac{\mathrm{d}}{\mathrm{d}}\frac{Ce(t)}{t} {}^* Cs(t) + Ce(t)^* \frac{\mathrm{d}}{\mathrm{d}}\frac{Cs(t)}{t} \tag{7}$$

Combining Eqs. (3), (4), and (7) results in the equation of motion of $Ce(t)$

$$\frac{\mathrm{d}}{\mathrm{d}}\frac{Ce(t)}{t} = \{Wg\}\ {}^* Ce(t) - Ce(t)^*\ \{Ws\} \tag{8}$$

III. Solutions and Attitude Trajectory Independence

The solution to the equation of motion [Eq. (8)] may be obtained as follows.
since

$$\{Ws\} = Cm^{-1*}\ \{Wg\}\ {}^* Cm \tag{9}$$

if we define an intermediate DCM Cx such that

$$Ce(t) = Cx(t)\ {}^*\ Cm \tag{10}$$

remembering that Cm is a constant misalignment DCM between the G frame and the S frame, then $Cx(t)$ follows the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}}\frac{Cx(t)}{t} = \{Wg\}\ {}^*\ Cx(t) - Cx(t)\ {}^*\ \{Ws\} \tag{11}$$

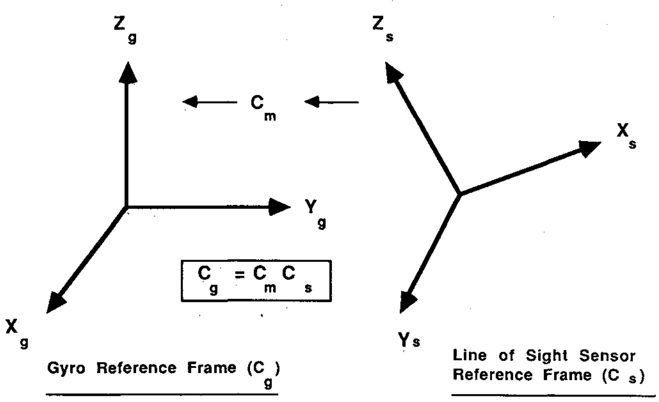

Fig. 1 Reference frame misalignment.

The solution of the foregoing equation is simply[1]

$$Cx(t) = Cg(t)\ {}^*\ Cx(0)\ {}^*\ Cg(t)^{-1} \tag{12}$$

where $Cx(0)$ is the initial condition of Cx, or in terms of the initial condition of Ce,

$$Cx(0) = Ce(0)\ {}^*\ Cm^{-1} \tag{13}$$

and the attitude knowledge error DCM at time t is

$$Ce(t) = Cg(t)^* Ce(0)^* Cm^{-1*} Cg(t)^{-1*} Cm \tag{14}$$

This proves the following theorem.

Theorem 1: The attitude knowledge error, induced by the gyro misalignment and the initial knowledge error, is a function of the attitude $Cs(t)$ [equivalently $Cg(t)$] only and is independent of the trajectory that the attitude is arrived at from the initial attitude at time $t = 0$ (provided no update to the attitude is made).

Appendix A shows the small angle approximation that lends its hand to easier geometrical interpretation.

IV. Misalignment Observation Using Simultaneous Three-Axis Attitude Updates

Simple and interesting properties may be derived from the solution as shown in Eq. (14). Suppose that the three axes' attitude knowledge error Ce is observable instantaneously at time $t = 0$ and is updated such that after the update, $Ce(0) = I$. (This is possible using two noncollinear LOS measurements.) Some time later at time t (and some attitude rotation was made during this time), the attitude error will be a function of the misalignment Cm, as

$$Ce(t) = Cg(t)^* Cm^{-1*} Cg(t)^{-1*} Cm \tag{15}$$

If the attitude rotation $Cs(t)$ and the unknown misalignment DCM Cm commute, i.e.,

$$[Cg\ (t),\ Cm] = [Cs\ (t),\ Cm] = 0 \tag{16}$$

where the commutator $[A, B]$ represents the commutation operation AB-BA, then

$$Ce(t) = I \tag{17}$$

where I is the unit matrix (null rotation matrix). This means that no attitude knowledge error will be measured. Although no attitude error can be measured, it does not mean that no misalignment information was observed. In fact, the commutation relation [Eq. (16)] provides an interesting relation between Cm and $Cg(t)$. It has been shown that the eigenrotations of Cm and $Cg(t)$ are collinear if neither Cm nor $Cg(t)$ are null rotations (see Ref. 5 or Appendix B). Nevertheless, the magnitude of rotation of Cm cannot be obtained.

For small misalignment when first-order approximation is used, the aforementioned statements may be proven rigorously in the following:

From Eq. (A3), the attitude errors at time t_1 and t_2 after the initial three-axis attitude update at time 0 are

$$e(t_1) = m - Cg(t_1)\, m \tag{18a}$$

$$e(t_2) = m - Cg(t_2)\, m \tag{18b}$$

and let

$$[I - Cg(t_1)]a_1 = 0 \tag{19a}$$

where I is the unit rotation matrix; then, a_1 is the eigenvector of the rotation $Cg(t_1)$. This proves that the matrix $[I - Cg(t_1)]$ has rank less than 3. Since $[I - Cg(t_1)]$ is the information matrix of the observation $e(t_1)$, m cannot be determined completely, with the only observation $e(t_1)$. It is possible that with another measurement at time t_2 such that

$$[I - Cg(t_2)]\, a_2 = 0 \tag{19b}$$

where the eigenvector a_2 is not collinear with a_1, and if we define $a_3 = a_1 x a_2 / ||a_1 x a_2||$, $a_1' = a_2 x a_3$, $a_2' = a_3 x a_1$, where x denotes vector cross-product operation, then using a new coordinate system with a_1, a_2', and a_3 or a_1', a_2, and a_3 as the triads, and express m in terms of the triads

$$m = k_1\, a_1 + k_2'\, a_2' + k_3\, a_3 \tag{20a}$$

$$m = k_1'\, a_1' + k_2\, a_2 + k_3\, a_3 \tag{20b}$$

Equations (18a) and (18b) may be rewritten as

$$e(t_1) = [I - Cg(t_1)]\, (k_2'\, a_2' + k_3\, a_3) \tag{21a}$$

$$e(t_2) = [I - Cg(t_2)]\, (k_1'\, a_1' + k_3\, a_3) \tag{21b}$$

and with the transformation from the $t = 0$ reference frame to that of new reference frames lead to dimension reduction transformations T_1 and T_2 such that

$$T_1\, e(t_1) = T_1{}^*\; [I - Cg(t_1)]^*\, \tilde{T}_1 \begin{bmatrix} k_2' \\ k_3 \end{bmatrix} \tag{22a}$$

$$T_2\, e(t_2) = T_2{}^*\; [I - Cg(t_2)^*\; \tilde{T}_2 \begin{bmatrix} k_1' \\ k_3 \end{bmatrix} \tag{22b}$$

where the expression $T_1{}^*[i - Cg(t_1)]^*\, \tilde{T}_1$ can be rewritten as

$$T_1{}^*[I - Cg(t_1)]^*\, \tilde{T}_1 = \begin{bmatrix} 1 - \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & 1 - \cos(\theta_1) \end{bmatrix} \tag{23}$$

after some manipulation. This shows, for a nontrivial rotation (i.e., $\theta_1 \neq 0$), that Eq. (23) has nonzero determinant and Eq. (22a), similarly Eq. (22b), may be inverted to solve for k_1', k_3 and k_2', k_3 pairwise. With the help of Eqs. (20a) and (20b), m may be completely determined (usually statistically for overdetermined k_3). Thus, it is proven that separating the misalignment and initial attitude error with this pair of observations is possible.

This leads to the following lemma.

Lemma 1: For small gyro reference frame misalignment calibration where simultaneous three-axis attitudes are available, it requires a minimum of three sets of three-axis attitude error measurements separated by two noncollinear rotations between the measurements.

Rigorous proof of this lemma for large misalignment error will be discussed in a future publication.

V. Observation Using an Infinitesimal Field-of-View, Line-of-Sight Sensor

In some spacecraft designs (e.g., Magellan), a simultaneous three-axis attitude update is not available because only a single LOS sensor is available.[2,3] Under this measurement environment, observation of the misalignment is difficult. In fact, when the field of view (FOV) of the LOS sensor diminishes, the observability of gyro misalignment about the LOS sensor boresight vanishes also.

Assume the attitude error is updated at time $t = 0$, because of the fact only a single LOS measurement is available, only the attitude error perpendicular to the LOS vector can be observed, and further, if we assume that there is no alignment error perpendicular to the LOS boresight, then at some time latrer, Eq. (13) represents the attitude error at this time t.[1] These assumptions, mathematically stated, are summarized as follows. For misalignment about the LOS of the sensor,

$$Cms = s \tag{24a}$$

where S is a known inertial LOS vector (e.g., a star vector) in the S frame. Similarly, for the attitude error solely about the LOS of the sensor (after the two-axis attitude error perpendicular to the LOS was updated), the measurement is

$$Ce(0)s = s \tag{24b}$$

which shows no error after the update. It is obvious that s is the eigenvector (axis) for both Cm and $Ce(0)$, and so

$$[Cm,\ Ce(0)] = 0 \tag{25}$$

$$Ce(0)s = Ce(0)^*Cms = Cx(0)s \tag{26}$$

Thus, at time t, when another LOS vector is observed (e.g., another star) denoted as s', we have

$$s = s' = v \tag{27}$$

in the S frame and v is the boresight vector in the S frame. Consequently, at time t, the measurement becomes

$$Ce(t)\, s' = Ce(t)^*Cmv = Cx(t)s \tag{28}$$

and so the measurements of attitude error and misalignment about the LOS direction transition as Cx, i.e., a single-composite rotation. And thus, the misalignment cannot be observed separately from the attitude error at time 0.

For small attitude error and small misalignment, the foregoing discussion may be proven in more rigor and is shown in the following.

Using Eq. (A3), assume that the measurement of attitude error was made at time $t = 0$, such that the attitude errors perpendicular to the boresight axis (e.g., x axis), i.e., e_y and e_z, are zeros; then,

$$\begin{bmatrix} e_x(t_1) \\ ey(t_1) \\ ez(t_1) \end{bmatrix} = Cs(t_1) \begin{bmatrix} e_x(0) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} I - Cs(t_1) \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \tag{29}$$

Let $Cs(t_1)$ be a 90 deg rotation about the LOS sensor boresight; then,

$$e_y(t_1) = m_y + m_z \tag{30a}$$

$$e_z(t_1) = -m_y + m_z \tag{30b}$$

which can be solved for m_y and m_z.

Another rotation may be taken; the equation of motion is now

$$\begin{bmatrix} e_x(t_2) \\ ey(t_2) \\ ez(t_2) \end{bmatrix} = Cs(t_2) \begin{bmatrix} e_x(0) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} I - Cs(t_2) \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \tag{31}$$

any measurement giving attitude errors perpendicular to the boresight, i.e., e_y, e_z, will give rise to the following:

$$e_y(t_2) = [Cs\ (t_2)]_{2,1}\ [ex\ (0) - m_x] \tag{32a}$$

$$e_z(t_2) = [Cs\ (t_2)]_{3,1}\ [ex\ (0) - m_x] \tag{32b}$$

where the subindices designate the elements of the respective matrix. Obviously, no independent evaluation of $e_x(0)$ and m_x is achievable and, thus, they are not independently observable.

This proves the following.

Lemma 2: Small gyro misalignment about an infinitesimal FOV LOS sensor cannot be observed at a later time, using the single infinitesimal FOV LOS sensor after an attitude rotation from the initial attitude, separately from the small attitude error about the same direction at the initial time.

From this, a proposition can be stated.

Proposition: For spacecraft design that requires inflight gyro reference frame calibration, more than one infinitesimal FOV LOS sensor with their boresights well separated (preferably at right angles) will be required. Rigorous proof of this lemma and the proposition for nonsmall angle misalignment and attitude error will be published in a future publication.

VI. Conclusion

This paper provides the fundamental theorems that enable design engineers to design the update system of any strapdown gyro application, with a proper understanding of the minimal number of update sensor requirements, without the havoc of trial and error. It shows that attitude error in a strapdown gyro application, because of the gyro misalignment, is a conservative function of only the attitude, between attitude updates. Furthermore, the calibration of the misalignment cannot be completed with a single small FOV LOS sensor. Moreover, it states a fundamental minimal maneuver requirement for the alignment calibration when multiple LOS sensors are available.

Appendix A: Small Angle Approximation

For small angle rotation, *Cx, Cm,* and *Ce* may be approximated to the first order by

$$Cx = I + \{x\} \tag{A1a}$$

$$Cm = I + \{m\} \tag{A1b}$$

$$Ce = I + \{e\} \tag{A1c}$$

Then, the solution [Eq. (13)] to the first-order approximation may be written as

$$\{e(t)\} = \{m\} + Cg(t)^*\ \{e(0) - m\}^* Cg(t)^{-1} \tag{A2}$$

This can be further simplified into a vector equation of *e, m*

$$e(t) = m + Cg(t)\ [e\ (0) - m] \tag{A3}$$

which is the same solution arrived at using the small angle rotation equation of motion

$$\frac{\mathrm{d}\ e(t)}{\mathrm{d}\ \ t} = \{Wg\}\ [e(t) - m] \tag{A4}$$

Appendix B: Commutative Direction Cosine Matrices

Direction cosine matrices form a representation group of all rigid-body rotations.[5] As we know, the members of the group are generally noncommutative. However, the members of the group with the same eigenaxis of rotation (the invariant vector of the rotation) form a subgroup with the members being commutative (customarily called Abelian group), i.e.,

$$AB = BA \tag{B1}$$

It can be proven that if Eq. (B1) is true, then A and B have the same eigenrotational axis.

Let a be a unit vector in an orthogonal three-space where the rotation matrices (or DCMs) are defined, and if a is the eigenvector of A, i.e.,

$$Aa = a \tag{B2}$$

then

$$BAa = Ba \tag{B3}$$

Using (B1), we obtain

$$ABa = Ba \tag{B4}$$

and let

$$b = Ba \tag{B5}$$

then

$$Ab = b \tag{B6}$$

Since A and B are orthogonal matrices, Eq. (B6) can only be true if

$$b = ka \tag{B7}$$

where k is a nonzero scalar.

Acknowledgment

This work was completed during the course of the Magellan attitudes and articulation control design and analysis under JPL Contract No. 956700. Acknowledgments are made to Dr. N.S. Reddy and Mr. Thomas Kelecy for their assistance in this work, and Messrs. R.P. Zietz and David Lukaszewski for a critical reading of the manuscript.

References

[1]Wertz, J. et al., *Attitude Determination and Control,* D. Reidel Publishing, Amsterdam, Holland, 1980.

[2]Huang, H. and Reddy, N.S., *"Magellan Gyro Error Parameter Inflight Calibration"* (to be published).

[3]Huang, H. and Reddy, N.S., *"Magellan Autonomous Attitude Determination and Attitude Updates and Gyro Parameter Calibrations,"* AIAA Paper 86-2042, Aug. 1986.

[4]Lipschutz, S., *Linear Algebra,* Schaum's Outline Series, McGraw-Hill, New York, 1968.

[5]Hammermesh, M., *Group Theory,* Addison-Wesley, Reading, MA, 1978.